CHAP I SYSTEMS OF LINEAR Equations and MATRICES

Consider the following equations:

a) $2x - y = 1$

b) $y - x^2 = 3$

c) $x + 2y + z = 1$

d) $z - x^2 - y^2 = 0$

e) $e^x + 2y + \sin(z) = 1$

f) $xy + 2z + 5y = 2$

Def. – A linear equ. in the variables $x_1, .., x_n$ is an equation that can be written as

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b \qquad (1)$$

where $a_1, .., a_n$ are real or complex numbers called the "coefficients" and b is also a real or complex number called the "independent term".

Solution

Consider the pairs: i) $(1,1)$, ii) $(2,1)$

Subst. in (a) leads to

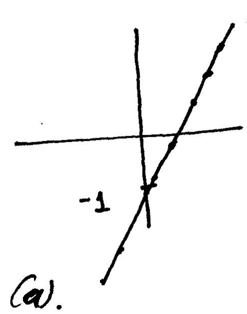
$(1,1)$: $2(1) - 1 = 1$ ✓ equ. is satisfied.

$(2,1)$: $2(2) - 1 = 3 \neq 1$ " is not ".

we will say that $(1,1)$ is a "soln" of equ. (a) and $(2,1)$ is not a "soln".

~~Ans Ques.~~ How many Solns. does equ. (a) have?

All points on the line $y = 2x - 1$ are solutions. In fact, they are all the solutions. We say that all points in L constitute the "general soln." of equ. (a).



Def. of Soln.

<u>Def.</u> — A soln. of a lin. equ. (1)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

is a list of n numbers $s_1, s_2, \ldots, s_n$ such that equ. (1) is satisfied when we substitute $x_1 = s_1, \ldots, x_n = s_n$.

LINEAR SYSTEMS.

Exampl 1.–
$$\begin{cases} 2x - y = 1 \\ x - y = -1 \end{cases}$$

(2,3) is a soln. Since

$2(2) - 3 = 1$ ✓

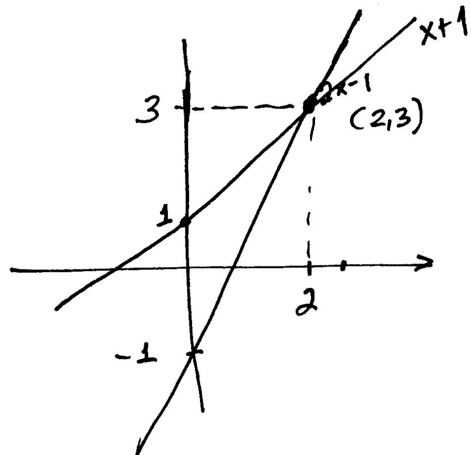
$2 - 3 = -1$ ✓

but (4,5) is not. Since

$2(4) - 5 = 3 \neq 1$

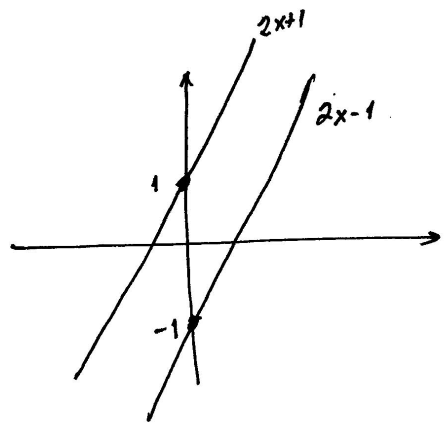
$4 - 5 = -1$ ✓

Geometrical soln.



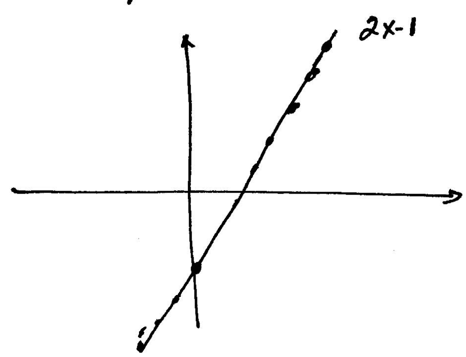
How may solns?

Example 2.–
$$\begin{cases} 2x - y = 1 \\ 2x - y = -1 \end{cases}$$



Example 3.–
$$\begin{cases} 2x - y = 1 \\ 4x - 2y = 2 \end{cases}$$

(2,3) and (3,5) are solns.
but (1,2) is not a soln.



Def. – A system of linear eqns. is a collection of one or more linear eqns. involving the same variables $x_1, x_2, .., x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdot\cdot + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdot\cdot + a_{mn}x_n = b_m \end{cases} \qquad (1)$$

m equations and n variables or unknowns.

Def. – A list of numbers $s_1, s_2, .., s_n$ is a soln. of system (1) if this list is a solution of every equation of system (1).

Examples 1 – 3 illustrate the following general result:

"Every system of linear eqns. has either no solns., exactly one soln or infinitely many solns."

<u>Def.</u>- A linear syst with one soln. or infinitely many solns is said to be consistent. Otherwise (no solns.), it's called inconsistent.

Matrix Representation

<u>Example 4.</u>-

$$\begin{cases} 2x_1 \qquad - 4x_3 = -10 \\ \qquad x_2 + 3x_3 = 2 \\ 3x_1 + 5x_2 + 8x_3 = -6 \end{cases}$$

Associated matrix to example 4 is given by

$$A = \begin{pmatrix} 2 & 0 & -4 \\ 0 & 1 & 3 \\ 3 & 5 & 8 \end{pmatrix}$$

It's called <u>Coefficient matrix</u>. Another important matrix is the <u>augmented matrix</u>:

$$A = \begin{pmatrix} 2 & 0 & -4 & -10 \\ 0 & 1 & 3 & 2 \\ 3 & 5 & 8 & -6 \end{pmatrix}$$

Solving a linear system

Replace given syst $\xrightarrow{by}$ another equivalent syst. easier to solve.

Three basic operations are performed on original syst.

1.- Scaling. Multiply by a nonzero constant
2.- Replacement. Add a multiple of one equation to another
3.- Interchange. Interchange two equations.

By an equivalent system we mean a new system that has the the same set of solns as the original one.

Work on transparencies example 4.

Example 4.-

$$\begin{aligned} 2x_1 \qquad - 4x_3 &= -10 \\ x_2 + 3x_3 &= 2 \\ 3x_1 + 5x_2 + 8x_3 &= -6 \end{aligned} \qquad \begin{pmatrix} 2 & 0 & -4 & -10 \\ 0 & 1 & 3 & 2 \\ 3 & 5 & 8 & -6 \end{pmatrix}$$

① Multiply 1^{st} equ. by $1/2$.

$$\begin{aligned} x_1 \qquad - 2x_3 &= -5 \\ x_2 + 3x_3 &= 2 \\ 3x_1 + 5x_2 + 8x_3 &= -6 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ \boxed{3} & 5 & 8 & -6 \end{pmatrix}$$

② Add $-3 \times$(equ. 1) to equ. 3 to eliminate x_1 coeff.

$$\begin{aligned} x_1 \qquad - 2x_3 &= -5 \\ x_2 + 3x_3 &= 2 \\ 5x_2 + 14x_3 &= 9 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 5 & 14 & 9 \end{pmatrix}$$

③ $-5 \times$ (equ. 2) + equ. 3

$$\begin{aligned} x_1 \qquad - 2x_3 &= -5 \\ x_2 + 3x_3 &= 2 \\ -x_3 &= -1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

4 $(-1)\times(\text{eq. }3)$

$$\begin{aligned} x_1 \quad -2x_3 &= -5 \\ x_2 + 3x_3 &= 2 \\ x_3 &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

5 $(-3)\times(\text{eq. }3) + \text{eq. }2$

$$\begin{aligned} x_1 \quad -2x_3 &= -5 \\ x_2 \quad &= -1 \\ x_3 &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

6 $2\times(\text{eq. }3) + \text{eq. }1$

$$\begin{aligned} x_1 \quad &= -3 \\ x_2 \quad &= -1 \\ x_3 &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Soln: $(x_1, x_2, x_3) = (-3, -1, 1)$.

Example 4.- (Summary)

$$
\begin{aligned}
2x_1 \qquad -4x_3 &= -10 \\
x_2 + 3x_3 &= 2 \\
3x_1 + 5x_2 + 8x_3 &= -6
\end{aligned}
\qquad
A = \begin{pmatrix} 2 & 0 & -4 & -10 \\ 0 & 1 & 3 & 2 \\ 3 & 5 & 8 & -6 \end{pmatrix}
$$

Basic operations or row operations

$$
\begin{aligned}
x_1 \qquad -2x_3 &= -5 \\
x_2 + 3x_3 &= 2 \\
x_3 &= 1
\end{aligned}
\qquad
B = \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}
$$

Further row operations

$$
\begin{aligned}
x_1 &= -3 \\
x_2 &= -1 \\
x_3 &= 1
\end{aligned}
\qquad
C = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}
$$

The three basic operations on the equations of the System Correspond to three elementary row operations on the augmented matrix:

1.- Multiply a row by a nonzero constant.

2.- Interchange two rows.

3.- Add a multiple of one row to another row.

Def.- We will Say that two matices are row equivalent if there is a sequence of elementary row operations that transform one matrix into the other.

Remark: Row operations are reversible

If the augmented matices of two linear systems are row equivalent then the two Systems have the same Solution set.

$10^{0.1}$

Exercise #12

$$
\begin{aligned}
&(1) \quad x_1 - 5x_2 + 4x_3 = -3 \\
&(2) \quad 2x_1 - 7x_2 + 3x_3 = -2 \\
&(3) \quad -2x_1 + x_2 + 7x_3 = -1
\end{aligned}
\qquad
\begin{pmatrix} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{pmatrix}
$$

① Eq (1) × (−2) + Eq (2) and Eq (1) × (2) + Eq. (3)

$$
\begin{aligned}
x_1 - 5x_1 + 4x_3 &= -3 \\
3x_1 - 5x_3 &= 4 \\
-9x_1 + 15x_3 &= -7
\end{aligned}
\qquad
\begin{pmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & -9 & 15 & -7 \end{pmatrix}
$$

② Eq (2) × (3) + Eq. (3)

$$
\begin{aligned}
x_1 - 5x_1 + 4x_3 &= -3 \\
3x_1 - 5x_3 &= 4 \\
\boxed{0 = 5}
\end{aligned}
\qquad
\begin{pmatrix} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & 4 \\ 0 & 0 & 0 & 5 \end{pmatrix}
$$

Contradiction!

Therefore, the System (1)-(3) is "<u>inconsistent</u>". It means, it has no solution.

Two Fundamental Questions About Linear Systems

1.- Is the System Consistent?

2.- Is the Solution unique?

Consistent means there is at least one Solution.

Unique means if a Solution exists that Solution is the only one.